

## THE SYZYGY VOLVELLE IN *ASTRONOMICUM CAESAREUM*

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**Abstract:** The theory and parameters behind a volvelle in Peter Apianus' *Astronomicum Caesareum* are investigated. It is found that he used the full Ptolemaic model of the Moon and also that he improved the layout of the volvelle by a clever trick.

**Keywords:** volvelle, Petrus Apianus, solar model, lunar model, syzygy

### 1 INTRODUCTION

In two earlier papers (Gislén, 2016; 2017) I studied Petrus Apianus' volvelles for eclipse duration and for planetary latitudes from his magnificent *Astronomicum Caesareum* (1540). Here I will study his volvelle for finding the true time of a syzygy (Figure 1). As it turns out it is possible to extract several pieces of interesting information from the volvelle.

### 2 THE VOLVELLE

In order to use the volvelle you need two input quantities, the solar anomaly (*argumentum solis*),  $\gamma_S$ , and the lunar anomaly (*argumentum lunae*),  $\gamma_M$ .

The volvelle has a circular rim with two graduations from 0° to 360°, one counter-clockwise and one clockwise, the first one with Arabic numbers and is to be used for lunar anomalies less

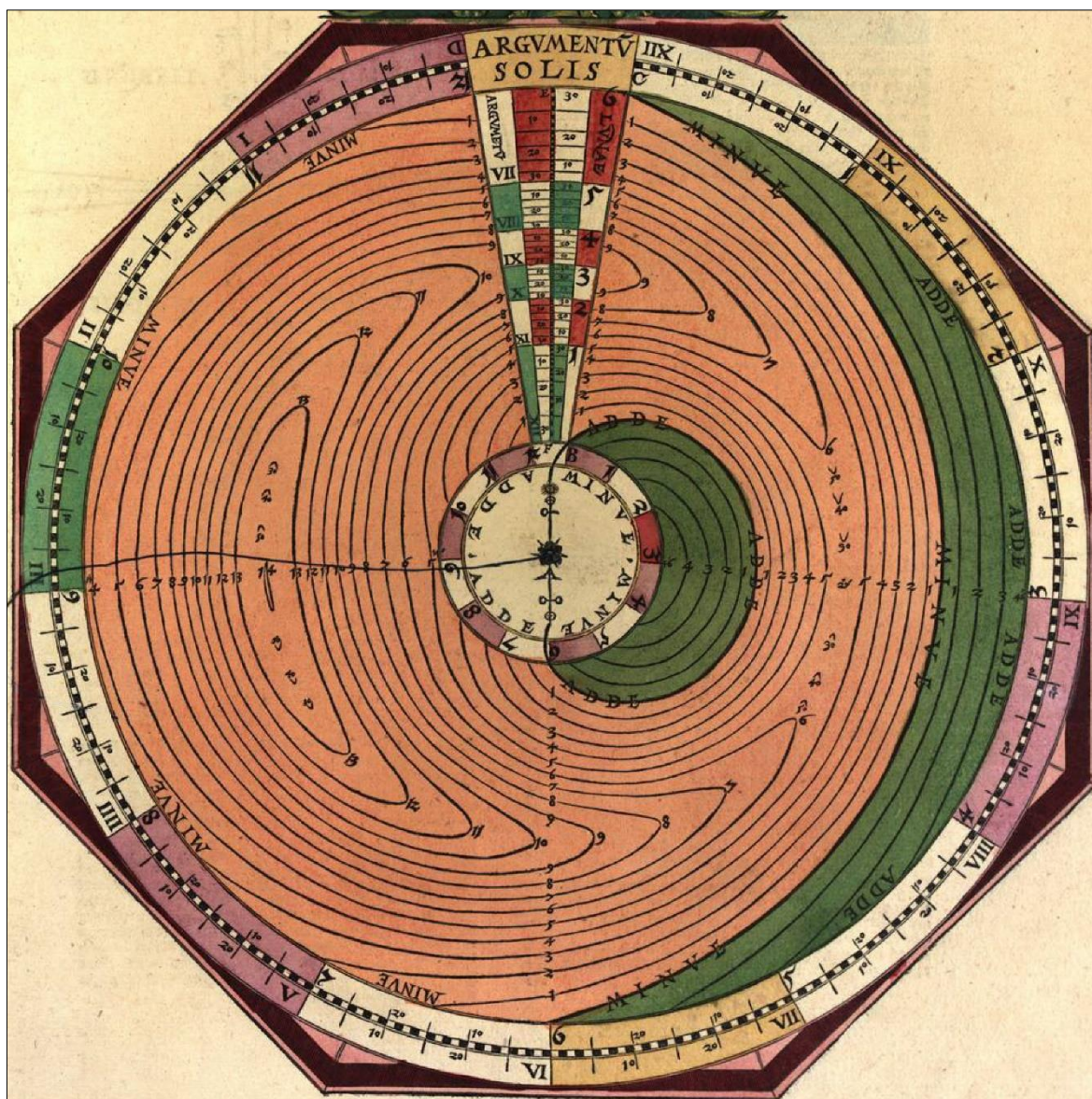


Figure 1: The volvelle.

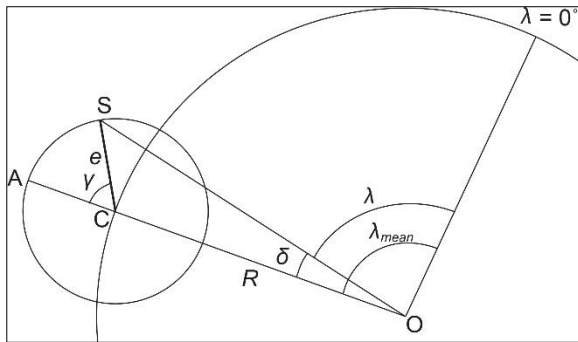


Figure 2: The solar model.

than 180°, the other one with Latin Numbers for anomalies larger than 180°. At the top of the volvelle there is a wedge-shaped area for setting the lunar anomaly. There is a thread coming from the centre of the volvelle with a small bead that can slide along the thread. You set the bead on the thread using the lunar anomaly scale, then align the thread against the solar anomaly on the rim and read off the correction in hours to the mean syzygy from the grid of isochronal lines below the bead. The reddish areas of the volvelle mean that the mean syzygy comes before the true one if the lunar anomaly is less than 180°, i.e. you have to add the correction to the mean syzygy time, for the greenish areas you have to subtract it. If the lunar anomaly is larger than 180° the colours have the opposite meaning.

### 3 THEORY

In the discussion below I use ‘velocity’ for ‘change in angle per time’. The unit of velocity will be arc minutes per hour.

In the Ptolemaic models (Neugebauer, 1975; Pedersen, 1974) used by Apianus, the true longitudes of the Sun and the Moon are given by

$$\lambda = \lambda_{mean} - \delta(\gamma) \tag{1}$$

Table 1: Solar Velocity.

Anomaly (°)	Solar Velocity (arcmins/hour)
0	2.37
10	2.38
20	2.38
30	2.38
40	2.39
50	2.40
60	2.42
70	2.43
80	2.44
90	2.46
100	2.48
110	2.49
120	2.51
130	2.52
140	2.54
150	2.55
160	2.55
170	2.56
180	2.56

where  $\lambda$  is the true longitude,  $\lambda_{mean}$  is the mean longitude,  $\delta(\gamma)$  the equation of centre, and  $\gamma$  the anomaly. At a mean syzygy, the mean longitudes are equal and the difference in longitude between the Moon and the Sun is then

$$\Delta\lambda = -\delta_M(\gamma_M) + \delta_S(\gamma_S) \tag{2}$$

where the indices  $M$  and  $S$  stand for Moon and Sun respectively.

If we divide this longitude difference by the difference in longitudinal velocities of the Moon and the Sun, the elongation velocity, we will get the time difference,  $\Delta T$ , between the true and mean syzygy. Thus

$$\Delta T = \Delta\lambda / (v_M - v_S) = (\delta_S(\gamma_S) - \delta_M(\gamma_M)) / (v_M(\gamma_M) - v_S(\gamma_S)) \tag{3}$$

The equations of centre and the velocities are given as tables in the Alfonsine Tables. However, it turns out that if one uses the velocity tables there, it is not possible to reproduce the volvelle. Below I will prove that Apianus used a more complicated model for the velocity of the Moon than that used for these tables.

#### 3.1 The Solar Model

For the Sun (Figure 2) the equation of centre is given by

$$\delta(\gamma) = \arctan(e \sin \gamma / (R + e \cos \gamma)) \tag{4}$$

This function is given as a table, *Equatio solis*, in the Alfonsine tables.  $\gamma$  is the anomaly of the Sun,  $R = 60$  and the Ptolemaic value of  $e$  is 2.5, but a least square fit to the data in that table gives  $e = 2.268$ .

We get longitudinal velocity of the Sun by taking the time derivative of (1):

$$v(\gamma) = v_{mean} \frac{e(R \cos \gamma + e)}{R^2 + 2Re \cos \gamma + e^2} v_\gamma \tag{5}$$

where  $v_\gamma$  is the mean velocity in anomaly. For the Sun, this velocity is the same as the mean solar velocity in longitude  $v_{mean} = 2.464'$ /hour. If we insert numbers in (5) we generate Table 1.

This table agrees within rounding errors with the corresponding table in the Alfonsine Tables.

#### 3.2 The Lunar Model

The Moon requires a considerably more complicated model (Figure 3), Ptolemy’s final lunar model.

O is the observer, M the Moon, C the mean centre attached to F by CF with fixed length  $R - s$ . A is the mean apogee, A’ the true apogee. The angle  $\eta$  is the elongation between the mean Moon and the mean Sun, the angle FOC being  $2\eta$ . The distance to the Moon is varied by a crank mechanism with F moving around the centre O with twice the elongation velocity. This complication was invented by Ptolemy to describe the lunar longitude at the quadrants. However, it causes the distances between the Moon and the Earth to

vary too much, the closest distance being  $\frac{2}{3}$  of the largest distance, something that could easily be detected by the naked eye by looking at the apparent size of the Moon. The red line OC points towards the mean Moon, OM points to the true Moon.

From the triangle OCN we get by standard trigonometry

$$c_3(2\eta) = \arctan(s \sin 2\eta / (\rho + s \cos 2\eta)) \quad (6a)$$

$$\rho(2\eta) = s \cos 2\eta + \sqrt{(R^2 - 2Rs + s^2 \cos^2 2\eta)} \quad (6b)$$

The function  $c_3$  is given in the Alfonsine Tables as *Equatio centri*.

The true anomaly  $\gamma'$  is given as a correction to the mean anomaly  $\gamma$  by  $\gamma' = \gamma + c_3$ .

The equation of centre is then

$$\delta(\gamma') = \arctan(r \sin \gamma' / (\rho + r \cos \gamma')) \quad (7)$$

This function is given in the Alfonsine Tables as *Equatio argumenti* with  $\rho = R = 60$ .

A least square fit to the data in the Alfonsine Tables gives  $r = 5.16$  and  $s = 10.317$ . The first quantity is different from the Ptolemaic value  $r = 5;25 = 5.417$ .

At syzygy ( $\eta = 0$ ) the change of  $\rho$  is zero and we can set  $\rho = R = 60$  in (6a).

We take the time derivative of (1) with the relations (6a) and (7) inserted and get, using the chain rule for derivation,

$$v(\gamma, \eta) = v_{mean} - \frac{r(R \cos \gamma' + r)}{R^2 + 2Rr \cos \gamma' + r^2} \left( v_\gamma + 2 \frac{dc_3}{d(2\eta)} v_\eta \right) \quad (8)$$

Here  $v_\gamma$  is the lunar mean velocity in anomaly and  $v_\eta$  the mean velocity in elongation.

We now consider the situation at syzygy.

Then  $\eta = 0$ ,  $c_3 = 0$ ,  $\gamma' = \gamma$ , and  $\frac{dc_3}{d(2\eta)} \Big|_{\eta=0} = \frac{s}{R+s}$  (9)

Using this we finally get

$$v(\gamma) = v_{mean} - \frac{r(R \cos \gamma + r)}{R^2 + 2Rr \cos \gamma + r^2} \left( v_\gamma + 2 \frac{s}{R+s} v_\eta \right) \quad (10)$$

Inserting numbers, the factor  $2s / (R + s) = 2 \cdot 10.317 / (60 + 10.317) = 0.2934$ . For Apianus it would be more natural to find this factor by using the table for  $c_3$  in the Alfonsine tables (*Equatio centri*). The difference between argument  $0^\circ$  and  $1^\circ$  in this table is  $0.1500^\circ$ , this divided by the difference in angle,  $1^\circ$ , is very nearly the derivative of  $c_3$  for  $\eta = 0$ , giving the factor above a value of 0.3. I have used this value for the calculations.

For the different velocities we have values derived from *Astronomicum Caesareum*, marginally different from the corresponding values in the *Almagest*:

- Mean lunar velocity in longitude,  $v_{mean} = 3.17639^\circ/\text{day} = 32.941'/\text{hour}$
- Mean lunar anomaly velocity,  $v_\gamma = 13.06499^\circ/\text{day} = 32.662'/\text{hour}$
- Mean elongation velocity,  $v_\eta = 12.19074^\circ/\text{day} = 30.477'/\text{hour}$

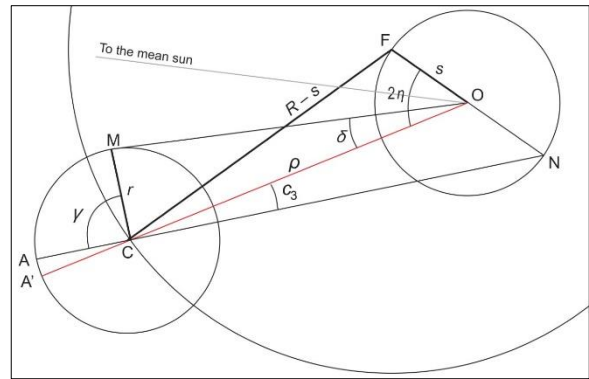


Figure 3: The lunar model.

Inserting these numbers, we can generate a table for the lunar velocity (see Table 2).

These are not the values we find in the Alfonsine Tables, and different editions of the Alfonsine Tables deviate slightly from each other. The values there are those we get if we set  $s = 0$  in the model above, i.e. a simpler model without the crank mechanism, equivalent to Ptolemy's first model of the Moon, presumably due to Hipparchos. As we will show in the next section, Apianus used the more complex model.

#### 4 CONFRONTING THE VOLVELLE

A close inspection of the volvelle reveals that at the inner right edge, for solar anomaly  $90^\circ$ , the time correction is written out as 4 hours 46 minutes ( $4;46 = 4.767$ ). Here the lunar anomaly is  $0^\circ$ . At the outer edge where the lunar anomaly is  $180^\circ$ , the correction is  $3;46 = 3.767$ . In both these cases the lunar equation is zero. In between these extremes, for lunar anomaly  $90^\circ$ , the time correction is  $5;27 = 5.45$ , with the opposite sign. On the left side of the volvelle, for solar anomaly  $270^\circ$ , there is, hardly legible, written a number 14 close to inner and the outer edges. The obvious interpretation is that at the inner edge the time correction is  $5 - 0;14 = 4;46$ , at the outer edge 4

Table 2: Lunar Velocity.

Anomaly( $^\circ$ )	Lunar velocity(arcmins/hour)
0	29.63
10	29.67
20	29.79
30	29.98
40	30.25
50	30.60
60	31.01
70	31.50
80	32.04
90	32.63
100	33.26
110	33.91
120	34.56
130	35.17
140	35.73
150	36.21
160	36.57
170	36.80
180	36.87

– 0;14 = 3;46, i.e. the same values as for solar anomaly 90° as to be expected from symmetry.

We now apply numerically the models above for these points in the volvelle and compute the time correction  $\Delta T$ . Along the line with solar anomaly 90°, the solar velocity in anomaly is constant and will be denoted  $v_S$  for reasons that will be apparent later. From the Alfonsine Tables we obtain that  $\delta_S(90) = 2.1658$ . The factor 60 below is to convert the  $\delta$  values from degrees to arc minutes.

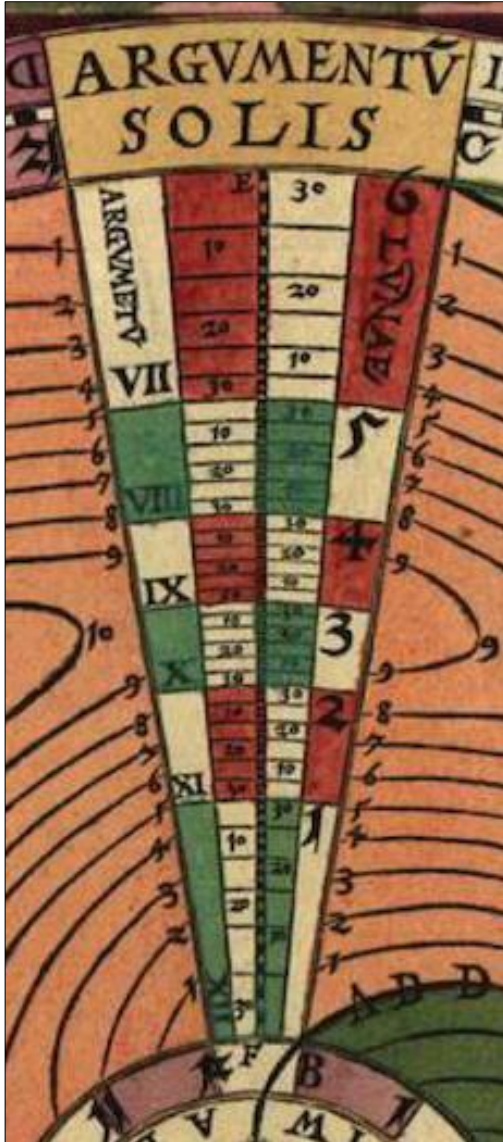


Figure 4: The lunar anomaly scale.

The inner edge:

$$\Delta T = 60 \cdot (\delta_S(90) - \delta_M(0)) / (v_M(0) - v_S) = 60 \cdot 2.1658 / (29.63 - v_S) = 4.767 \quad (11)$$

or  $29.63 - v_S = 27.26$ , thus  $v_S = 2.37$ .

The outer edge:

$$\Delta T = 60 \cdot (\delta_S(90) - \delta_M(180)) / (v_M(180) - v_S) = 60 \cdot 2.1658 / (36.87 - v_S) = 3.767 \quad (12)$$

or  $36.87 - v_S = 34.50$ , thus  $v_S = 2.37$ .

For  $\gamma_M = 90^\circ$ , we take  $\delta_M(90) = 4.9150$  from the Alfonsine Tables (*Equatio argumenti*).

$$\Delta T = 60 \cdot (\delta_S(90) - \delta_M(90)) / (v_M(90) - v_S) = 60 \cdot (2.1658 - 4.9150) / (32.63 - v_S) = -5.45 \quad (13)$$

or  $32.63 - v_S = 30.27$ , thus  $v_S = 2.36$ .

This is what is to be expected, we should get the same value for the solar velocity if the scheme works. What is not expected is that it is the value for solar anomaly 0°, not 90°. Apianus apparently used 2.36 as a generic constant for the solar velocity for all the points in the volvelle and that would simplify his calculations considerably. As the solar anomaly velocity is almost constant, the error introduced would be small, at most a couple of minutes. However, I think a more natural choice would have been to use the mean anomaly velocity.

It is interesting to note that the three elongation velocities above, 27.26, 30.27, and 34.50, agree quite well with the corresponding (less accurate) velocities derived in my earlier paper (Gislén, 2016) on Apianus' eclipse volvelle: 27.38, 30.32, and 34.43.

If we now use this scheme, it turns out that we can generate time correction values that very precisely correspond to the isochronal lines of the volvelle. Apianus gives two examples of using the volvelle for syzygies that can be used as a further check of the scheme, one New Moon on 14 February 1500, associated with the birth of the Holy Roman Emperor Charles V and a Full Moon on 25 February 1503 associated with the birth of his brother Ferdinand I.

Example 1. New Moon of 14 February 1500.

$$\gamma_S = 8 \text{ signs } 1^\circ 44' = 241.73^\circ, \gamma_M = 2 \text{ signs } 14^\circ 47' = 74.78^\circ$$

Apianus gives the time correction as 13;23. This is certainly a calculated value; it is not possible to get a time with this precision using the volvelle. The scheme above gives 13;26.

Example 2. Full Moon of 25 February 1503.

$$\gamma_S = 8 \text{ signs } 13^\circ 42' = 253.70^\circ, \gamma_M = 4 \text{ signs } 16^\circ 34' = 136.37^\circ$$

Apianus gives the time correction as 10;19. The scheme above gives 10;21.

For those interesting in experimenting with the model there is a downloadable Java application PASyzygy.jar on my web site; see <http://home.thep.lu.se/~larsg/Site/Welcme.html>

## 5 THE DISTORTED LUNAR ANOMALY SCALE

The lunar anomaly scale (Figure 4) in the wedge at the top of the volvelle is distorted such that the central part is contracted and the other parts extended. What is the purpose of this?

I think the reason was to try to make the isochronal lines of the volvelle more equidistant. This would make interpolation in the volvelle much easier both for its construction and for the later use. If we move radially in the volvelle for fixed solar anomaly  $\gamma_S = 0$ , increasing the lunar anomaly, the time correction is approximately determined by the lunar equation  $\delta(\gamma_M)$ . Other solar anomalies would essentially only add a constant to this function. A qualitative graph of  $\delta(\gamma_M)$  is shown in Figure 5a with the anomaly  $\gamma_M$  on the horizontal axis.

If we now consider this as a picture of a ‘hill’, a set of altitude curves in a map of the hill would not be equidistant, they would spread out as we approach the top of the hill. In Figure 5a I have mirrored the right part of the black curve from the point where the tangent to the original curve is horizontal, to generate the red curve, i.e. I have reversed the slope of the curve where the slope is negative. Suppose now that we unite the left, black part of the curve and the red right part creating a monotonous rising function and use this curve as a conversion curve from the anomaly coordinate  $\gamma_M$  on the horizontal axis to a plotting scale coordinate  $P$  on the vertical axis, Figure 5b. If we then replot the original function as a function of  $P$ , it can be proven mathematically that the graph of original function will be reduced to straight lines, in this case the curves in Figure 5c. This means that if we do this for the volvelle, the isochronal lines for the replotted ‘hill’ would be equidistant, at least locally.

I measured the position in pixels for each 10' step of the actual lunar anomaly scale in the picture of the volvelle. I then used the table of the lunar equation to construct a conversion function between lunar anomaly and an ideal plotting scale, using the procedure described above. Finally, I rescaled this ideal scale with a factor such that the largest item, for  $\gamma_M = 180^\circ$ , had the same size as the total length of the measured anomaly scale, 668 pixels. Figure 6 shows a comparison between the actual measured (blue) anomaly scale and the constructed ideal plotting scale (red). The agreement is quite good and indicates that the distorted scale really had the intention of making the isochronal curves in the volvelle more equidistant. Actually, this trick is also used in the Venus latitude volvelle, although in that case it does not seem to be quite necessary.

## 6 CONCLUDING REMARKS

The syzygy volvelle confirms the reputation that Petrus Apianus had as being one of the most famous astronomers of the sixteenth century. He works here with a complicated scheme and makes clever approximations and simplifications where they can be made, in order to produce a pedagogical and elegant instrument. It is ironic that

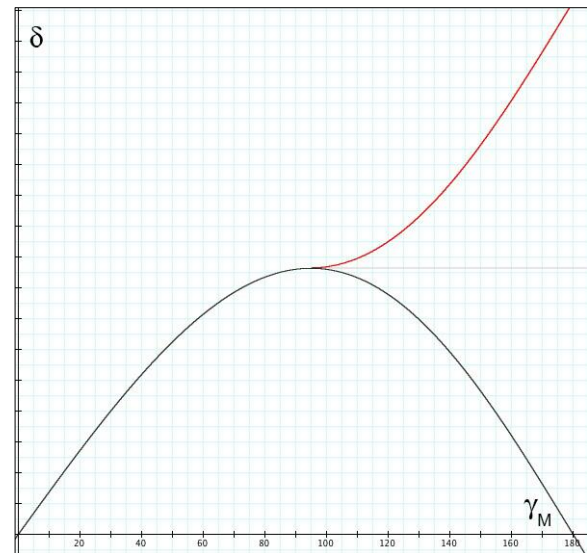


Figure 5a: Scale transformation.

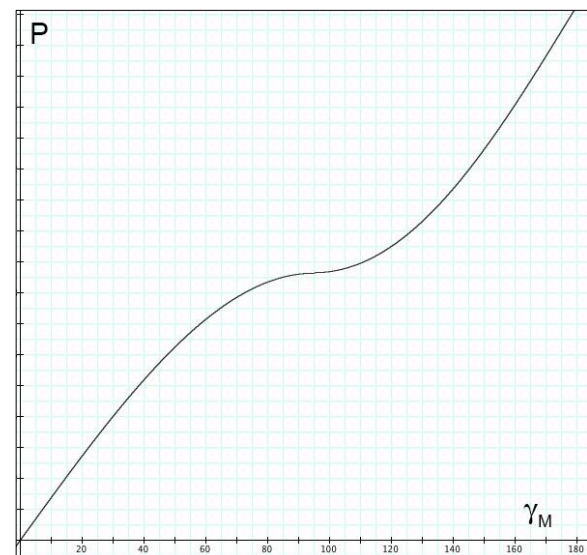


Figure 5b: The conversion function.

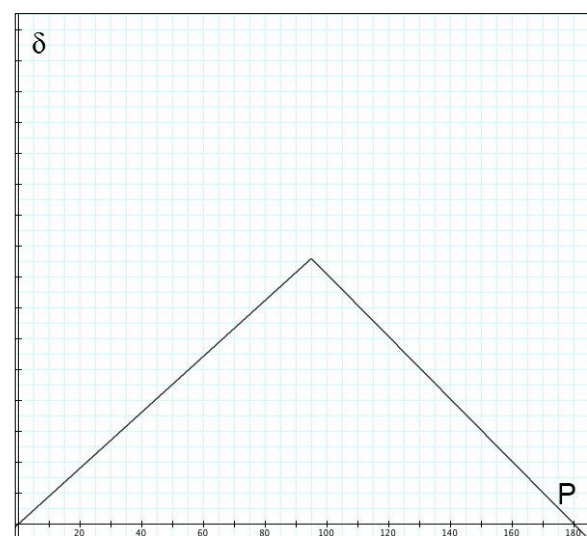


Figure 5c: The replotted curve.

his magnificent *Astronomicum Caesareum*, based on the Ptolemaic model, soon after its publication would become obsolete in the new astro-

nomical era based on the heliocentric model, with Copernicus' *De revolutionibus* (1543), published just three years after *Astronomicum Caesareum*.

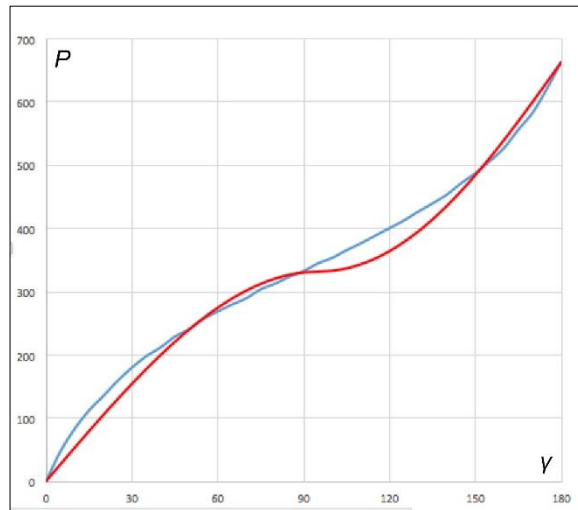


Figure 6: Comparison of the conversion functions.

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