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PERIPHERIES OF EPICYCLES IN THE *GRAHALĀGHAVA*

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Abstract: For finding the true positions of the Sun, the Moon and the five planets the Indian classical astronomical texts use the concept of the *manda* epicycle which accounts for the equation of the centre. In addition, in the case of the five planets (Mercury, Venus, Mars, Jupiter and Saturn) another equation called *śīghraphala* and the corresponding *śīghra* epicycle are adopted. This correction corresponds to the transformation of the true heliocentric longitude to the true geocentric longitude in modern astronomy. In some of the popularly used handbooks (*karaṇa*) instead of giving the mathematical expressions for the above said equations, their discrete numerical values, at intervals of 15°, are given.

In the present paper using the data of discrete numerical values we build up continuous functions of periodic terms for the *manda* and *śīghra* equations. Further, we obtain the critical points and the maximum values for these two equations.

Keywords: Equation of the centre, epicycle, periphery, apogee, perigee, equation of the conjunction, *śīghraphala, mandaphala, paridhi, Grahalāghava,* Gaṇeśa Daivajña

1 INTRODUCTION

The *Grahalāghava* (*GL*) is one of the most popular *karaṇa* texts of Indian astronomy, and was written by the famous sixteenth-century author Gaṇeśa Daivajña. After Bhāskara-II of the twelfth century there was a decline for a brief period in the development of mathematics and astronomy in India. But we see tremendous work was done in the south i.e., in Kerala and Maharashtra, giving rise to some of the great and eminent luminaries like Nilakaṇṭa Somayājī and Gaṇeśa Daivajña.

Gaṇeśa Daivajña is unique because he dispensed with trigonometric terms in his computations and replaced them with suitable algebraic approximations. This method helped many almanac (*pañcāṅga*) makers to do calculations in a simple way. So even today, the *GL* is one of the popular texts among almanac-makers.

The text of the *GL* consists of 187 verses (*ślokas*) distributed in 14 chapters. In chapters 2 and 3 the true positions of the Sun, the Moon and the five planets are discussed. For the Sun and the Moon there is only one correction, namely the *mandaphala*, which corresponds to the equation of the centre, taking into account the eccentricity of the body's orbit. But for the five planets, apart from the *mandaphala* one more equation called *śīghraphala* is applied. *Śīghraphala* converts heliocentric position to geocentric position of the planets. In order to determine the two equations *manda* and *śīghra,* Gaṇeśa Daivajña gives discrete values, called *mandāṅkas* and *śīghrāṅkas.* These are obtained by multiplying the actual *manda* and *śīghra* corrections by 10. Further, these values are in arc minutes (*kalās*), and given in integers for every 15°. Gaṇeśa Daivajña does not provide either the peripheries (*paridhis*) of the epicycles nor does he mentions explicitly the expressions for the two equations. However, in the case of the Sun and the Moon he gives explicit approximate algebraic expressions for the equation of the centre. In this paper we estimate the ranges of peripheries of the equations for each of the bodies.

2 THE METHOD OF THE *GRAHALĀGHAVA* FOR THE EQUATION OF THE CENTRE

In obtaining the mean positions of the Sun and the Moon it was earlier assumed that these bodies moved in circular orbits around the Earth with uniform angular velocities. However, observations revealed that the motions were non-uniform. The true positions were related to the epicyclic theory that is explained in the following section.

2.1 Epicyclic Theory and the Equation of the Centre

The theory is that while the mean Sun or the Moon move along a big circular orbit (see Figure 1), the actual Sun and Moon move along a smaller circle called an epicycle, whose centre is on the larger circle.

The larger circle *ABP* with the Earth *E* as its centre is called the deferent circle (*kakṣāvṛtta*). Let *A* be the position of the mean Sun when the true Sun is farthest from the Earth. The line *AEP* is called the apse line and *AE* is the radius (*trijyā*) of this orbit. The epicycle, with *A* as centre and a prescribed radius (smaller than *AE*) is called the *nīcoccavṛtta.* Let the apse line *PEA* cut the epicycle at *U* and *N.* The two points *U* and *N* are respectively called the apogee (*mandocca*) and the perigee (*mandanīca*) of the Sun. Note that as the Sun moves (as seen from Earth) along the epicycle, the Sun is farthest from the Earth at *U* and nearest at *N*.

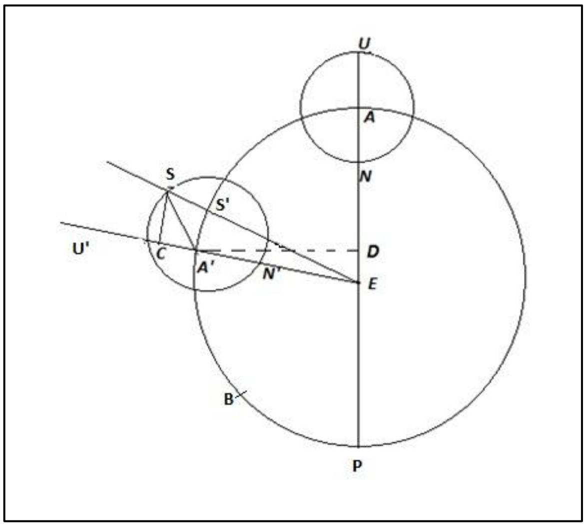

Figure 1: Epicyclic theory.

The epicyclic theory assumes that as the centre of the epicycle (i.e. mean Sun) moves along the circle *ABP* in the direction of the signs of the zodiac (from west to east) with the velocity of the mean Sun, the true Sun itself moves along the epicycle with the same velocity but in the opposite direction (from east to west). Further, the time taken by the Sun to complete one revolution along the epicycle is the same as that taken by the mean Sun to complete a revolution around the orbit.

Now in Figure 1, suppose the mean Sun moves from *A* to *A'*. Let *A' and E* be joined, cutting the epicycle at *U'* and *N'*, which are the current positions of the apogee (*mandocca*) and the perigee (*mandanīca*). While the mean Sun is at *A'*, suppose the true Sun is at *S* on the epicycle so that *U'Â'S* = *U'ÊA*. Join *ES*, cutting the orbit (i.e., circle *ABP*) at *S'*. Then *A'* is the mean Sun (*madhya Ravi*) and *S'* is the true Sun (*spaṣṭa* or *sphuṭa Ravi*). The difference between the two positions viz., *A'ÊS'* (or arc *A'S'*) is called the equation of the centre (*mandaphala*).

In order to obtain the true position of the Sun, it is necessary to get an expression for the equation of the centre which will have to be applied to the mean position.

In Figure 1 *SC* and *A'D* are drawn perpendicular to *U'N'E* and *UNE* respectively. The arc *AA'* (or *AÊA'*), the angle between the mean Sun and the apogee, is called the mean anomaly (*mandakendra*, henceforth *MK*) of the Sun.

We have, in the right-angled triangle *A'DE,*

$\sin A\hat{E}A' = \sin D\hat{E}A' = A'D/A'E$

so that, $A'D = R\sin AA' = R\sin MK$ is called *R* sine of anomaly (*mandakendrajyā*), where $R = A'E$ and MK = arc AA'.

From the similar right-angled triangles *SCA'* and *A'DE*, we have

$SC/SA' = A'D/AE'$

and

$SC = (SA' \times A'D)/A'E$

Since *SA'* and *A'E* are respectively the radii of the epicycle and the orbit, these are proportional to the circumferences of the two circles; that is

$SA'/A'E$ = circumference of the epicycle/ circumference of the orbit

∴ SC = (circumference of the epicycle/ circumference of the orbit) × $A'D$

Taking the circumference of the orbit as 360°, we have

SC = (circumference of the epicycle × *mandakendrajyā*)/360°

Now, taking *SC* approximately the same as *A'S'*, the equation of the centre (*mandaphala,* henceforth *MPH*) is given by

$R\sin(MPH)$ = circumference of the epicycle × *mandakendrajyā*)/360°

$= (p/R) \times R\sin MK$

i.e. $\sin(MPH) = (p/R) \times \sin MK$

where $R\sin MK$ is the 'Indian sine' (*jyā*)[1] of the anomaly *MK* of the Sun. The maximum value of the equation of the centre, i.e., $\sin(MPH)$ is p/R (in radians) or $p/2\pi$ (in degrees).

In his *Grahalāghava*, Gaṇeśa Daivajña gives the following verse to obtain the anomaly from the apogee (*mandakendra*) of the planet:

> If the *bhuja* (of the *manda* anomaly) is less than three *rāśis* (signs) then take that itself, if the anomaly is greater than three *rāśis* and less than six *rāśis* then consider the difference of six *rāśis* (180°) and the anomaly as the *bhuja*, if the anomaly is greater than six *rāśis*

and less than nine *rāśis* then subtract six *rāśis* (180°) from the anomaly to get the *bhuja* and if the anomaly is greater than nine *rāśis* and less than twelve *rāśis* then the remainder of subtracting it from twelve *rāśis* (360°) is the *bhuja.* (*Grahalāghava*, Ch-II, *śloka* -1; our English translation).

This means the anomaly from the apogee (*mandakendra*, *MK*) = apogee (*mandocca*) of the planet – Mean planet. *MK* is expressed as an acute angle; to get this, we use the following procedure:

(1) If 0° ≤ *MK* < 90° then *MK* itself is the argument (*bhuja*) i.e., *bhuja* = *MK.*
(2) If 90° ≤ *MK* < 180° then *bhuja* = 180° – *MK*
(3) If 180° ≤ *MK* < 270° then *bhuja* = *MK* – 180°
(4) If 270° ≤ *MK* < 360° then *bhuja* = 360° – *MK*

According to the *Grahalāghava*, the apogees of the heavenly bodies are as shown in Table 1.

It is assumed that the apogee of the Moon varies, whereas those of the other bodies are fixed.

The method of finding the equation of the centre of the Sun is explained in the following verse*:*

The difference between the *mandocca* (apogee) and the mean planet is called (*manda*) *kendra* (anomaly). If the *kendra* is within six *rāśis* from *Meṣa* or within six *rāśis* from *Tulā*, (correspondingly) the *mandaphala* (the equation of the centre) is positive or negative.

In the case of *Ravi* (Sun), divide the *bhuja* (of the *mandakendra*) by 9, subtract it from 20 and multiply the result by itself; (this is the numerator). Divide the numerator by the difference between 57 and one-ninth of the numerator. (*Grahalāghava*, Ch-II, *śloka* -2; our English translation).

This means, find the anomaly from the apogee (*MK*) of the Sun and express *MK* in terms of *bhuja* of *MK* as explained earlier. Denote *bhuja* of *MK* by *BMK*.

(1) Subtract (*BMK*/9) from 20 and multiply this by (*BMK*/9).
(2) Divide the result of (1) by 9.
(3) Subtract the result of step (2) from 57.
(4) Express the results of step (3) and step (1) in seconds of arc (*vikalās*) and divide the result of step (1) by that of step (3).

Then the result is the equation of the centre of the Sun.

i.e., The equation of the centre of the Sun =
$[20 - (BMK/9)] \times (BMK/9)\ /\ [57 - \{(20 - (BMK/9)) \times (BMK/9)\ /\ 9\}]$

Note:

(1) In devising the above equation the author dispenses with the trigonometric ratio sine.

(2) If the anomaly from the apogee is within 6 signs from Aries (*Meṣa)* (i.e., 0° < *MK* < 180°) then the equation of the centre is additive.
(3) If the anomaly from the apogee is within 6 signs from Libra (*Tulā*) (i.e., 180° < *MK* < 360°) then the equation of the centre is subtractive.
(4) If the anomaly is 0° or 180° then the equation of the centre is zero.

2.2 Rationale for the Equation of the Centre of the Sun

Śrīpati Bhaṭṭa's (ca. tenth century) expression for the *R* sine (*jyā*) of the anomaly is as follows:

Subtract the *manda* anomaly from 180 and multiply by itself; (this is the numerator). Divide the numerator by the difference between 10125 and one-fourth of the numerator. (Finally) thus obtained result is multiplied by 120 to get the *jyā* (*R* sine) of the *manda* anomaly of the Sun. (*Siddhānta-śekhara*, Ch-III, *śloka*-17; our English translation).

This implies the anomaly from the apogee (*MK*) in degrees is subtracted from 180° and the remainder is multiplied by the same quantity (MK). Then the result is divided by its one-fourth, subtracted from 10125. This result is multiplied by twice sixty (i.e., by 120).

Table 1: Apogee of the heavenly bodies.

Body	Apogee
Sun	78°
Mars	120°
Mercury	210°
Jupiter	180°
Venus	90°
Saturn	240°

i.e. In symbols, R sine of anomaly = $[(180 - MK)MK \times 120]\ /\ \{10125 - [(180 - MK)/4] \times MK\}$

where *MK* stands for the *bhuja* of the anomaly

i.e., R sine $(MK) = [(180 - MK)MK \times 480]\ /\ [40500 - (180 - MK)MK]$

$= \{[(180 - MK)\ /\ 9]\ [(MK/9) \times 480]\}\ /\ \{[405000/(9 \times 9)] - [(180 -- MK)/9](MK/9)\}$

(dividing by 9 × 9)

$= \{[20 - (MK/9)][MK/9] \times 480]\}\ /\ \{500 - [20 - (MK/9)](MK/9)\}$ (1)

The above derivation is based on the significant and unique formula of Bhāskara I (c. 629 CE);

i.e., $\sin\theta = [4\,(180° - \theta)\,\theta]\ /\ [40500 - (180° - \theta)\,\theta]$

Now, according to the *Grahalāghava* the maximum equation of the centre (*parama mandaphala*) of the Sun

= (125°/57) ≈ 2° 11′ 34″.

Table 2: The Sun's equation of the centre, *MPH* and *manda* periphery, *p*

MK	*MPH*	*Manda periphery (p)*
15°	0.570	13°.834
30°	1.093	13°.735
45°	1.541	13°.69
60°	1.886	13°.68
75°	2.104	13°.689
90°	2.179	13°.692

∴ The equation of the centre of the Sun = (125°/57) × (*mandakendrajyā*)/120)

= [125 / (57 × 120)] × {[(20 – (*MK*/9)(*MK*/9) × 480] / 500 – [20 – (*MK*/9)(*MK*/9)]} Using (1)

= {(125 / (57)[(20 – (*MK*/9)(*MK*/9) × 4]} / {500 – [20 – (*MK*/9)](*MK*/9)}

= {(500 / (57)[(20 – (*MK*/9)(*MK*/9)]} / {500 – [20 – (*MK*/9)](*MK*/9)}

= {[(20 – (*MK*/9)](*MK*/9)} / {[500/(500/57)] – [20 – (*MK*/9)](*MK*/9) / (500/57)}

= {[(20 – (*MK*/9)](*MK*/9)} / {57 – [20 – (*MK*/9)](*MK*/9) / 8.771928}

i.e., Equation of the centre of the Sun

≈ {[(20 – (*MK*/9)](*MK*/9)} / {57 – [20 – (*MK*/9)] (*MK*/9) / 9]}

The exact formula for the equation of the centre of the Sun is $\sin^{-1}[(p/R)\sin MK)$ where R = 360°, p is the periphery of the *manda* epicycle (in degrees) and *MK* is the Sun's anomaly (from the *apogee*, *mandocca*).

Using this formula with the range of *MK* from 15° to 90° the Sun's equation of the centre, *MPH*, and the periphery (*paridhi*) of the *manda* epicycle, *p*, are estimated and listed in Table 2.

In order to estimate the *manda* periphery of the Sun from 0° to 90°, we adopt the formula $p = A + B \sin(MK)$. The related procedure is explained in later sections. The periphery of the Sun for *MK* = 0° is 14°.001 and for *MK* = 90° is 13°.692.

Similarly, the equation of the centre of the Moon is given in the following verse

> In the case of *Vidhu* (Moon), one-sixth of the *manda* anomaly is subtracted from 30 and the remainder is multiplied by the same; (this is the numerator). This numerator is divided by the difference between 56 and one-twentieth of the numerator. This is Moon's equation of the centre. (*Grahalāghava,* Ch-II, *śloka* -3; our English translation).

Table 3: The Moon's equation of the centre, *MPH* and *manda* periphery, *p*

MK	*MPH*	*Manda periphery (p)*
15°	1.307	31°.752
30°	2.512	31°.573
45°	3.547	31°.526
60°	4.347	31°.544
75°	4.854	31°.576
90°	5.027	31°.591

This can be expressed as the following formula:

Equation of the centre of the Moon = {[30 – (*MK*/6)](*MK*/6)} / {56 – [30 – (*MK*/6)(*MK*/6) / 20]}

2.3 Rationale for the Equation of the Centre of the Moon

We have *R* sine of anomaly = [(180 – *MK*)*MK* × 480)] / [40500 – (180 – *MK*)*MK*]

According to Śrīpati Bhaṭṭa, dividing the numerator and the denominator by 6 × 6,

*R*sin(*MK*) = {[(180 – *MK*)/6)]*MK* × (480/6)} / {(40500/ 6 × 6) – [(180 – *MK*/6)](*MK*/6)}

= {(30 – *MK*/6)(*MK/6*) × 480} / {120 × [1125 – [30 – (*MK*/6)](*MK*/6)} (2)

According to the *Grahalāghava* the maximum equation of the centre of the Moon = 5°.

∴ Equation of the centre of the Moon = (5 × *R* sine of anomaly) / 120

= {5 × [30 – (*MK*/6)](*MK*/6) × 480} / {120 × [1125 – [30 – (*MK*/6)](*MK*/6)]} using (2)

= {(2400/120) [30 – (*MK*/6)](*MK*/6) } / {[1125 – [30 – (*MK*/6)](*MK*/6)]}

= {20[30 – (*MK*/6)](*MK*/6)} / {[1125 – [30 – (*MK*/6)](*MK*/6)]}

= {[30 – (*MK*/6)](*MK*/6)} / {(1125/20)[30 – (MK/6)](MK/6) / 20}

= {[30 – (*MK*/6)](*MK*/6)} / {56.25 – [(30 – *MK*/6)](*MK*/6) / 20}

i.e., Equation of the centre of the Moon ≈ {[30 – (*MK*/6)](*MK*/6)} / {56 – [(30 – *MK*/6)(*MK*/6)]/ 20}

In the similar way as in the case of the Sun's periphery, the Moon's periphery is estimated and listed in Table 3.

The periphery of the Moon for *MK* = 0° is 32°.075 and for *MK* = 90°, it is 31°.591.

3 EQUATION OF THE CENTRE OF THE PLANETS

In the case of the five planets in the *GL*, instead of providing direct expressions, Gaṇeśa Daivajña gives discrete numerical values for the equation of the centre (*mandaphala*) in degrees at intervals of 15° of the *manda* anomaly. He has multiplied the equation of the centre by 10 (to avoid fractions) and calls them as *mandāṅkas,* as given in Table 4.

In order to estimate the underlying *manda* per-

Table 4: Discrete values of the equation of the centre (*mandāṅkas*) of the planets.

Planets	15°	30°	45°	60°	75°	90°
Mars	29	57	85	109	124	130
Mercury	12	21	28	33	35	36
Jupiter	14	27	39	48	55	57
Venus	06	11	13	14	15	15
Saturn	19	40	60	77	89	93

ipheries of the different planets, we adopt the following two procedures:

(1) As a first approximation, the

Equation of the centre $(MPH) = (p/R)\sin(MK)$ in radians (3)

$\therefore\ p = (MPH \times R)\ /\ \sin(MK)$ in degrees. (4)

(2) As the second approximation, or the correct expression

$\sin(MPH) = (p/R)\sin(MK)$ in radians (5)

where p is periphery of the epicycle, MK is the *manda* anomaly and R is 2π radians or 360°.

As an example, based on equation (4) the *manda* periphery (p) of Mars is given in Table 5.

We find from Table 5 that the *manda* periphery increases from 70°.40145 to 81°.68142 as the *manda* anomaly (MK) increases from 15° to 90°.

Note: The *manda* periphery for MK = 0 cannot be obtained from equation (4) since the denominator vanishes.

Now since p varies from 70°.40145 to 81°.68142, we express the periphery p for any given MK in the form

$p = A + B\sin(MK)$ (6)

for which we have to determine the constant coefficients A and B. Tentatively, for MK = 30° and 90°, we get the respective linear equations as

$p = A + (B/2)$ and $p = A + B$ (7a)

Solving these equations, we obtain A = 61°.5752 and B = 20°.10622. (It is to be noted that we do not get the same values of A and B as above if we consider the other pairs of the linear equations.)

This means that for the above values of A and B, periphery p varies from 66°.77908 to 81°.68142 as MK varies from 15° to 90° in the case of Mars. Similarly, estimating the *manda* peripheries for the other four planets namely, Mercury, Venus, Jupiter and Saturn, we get the values as shown in Table 6.

When MK = 0°, formula (6) becomes $p = A$ hence the above table of *manda* peripheries can be now listed for MK = 0° to 90° by solving equations (7) by finding the A and B values.

Now, considering the actual expression for the equation of the centre given by equation (5) we have

$\sin(MPH) = (p/R)\sin(MK)\ \Rightarrow$
$p = [R \times \sin(MPH)]\ /\ \sin(MK)$ (7b)

Following the same procedure as for Mars in the case of the remaining four planets we get the *manda* peripheries as shown in Table 7.

From Table 8, we find that the *manda* periphery 'p' increases as anomaly MK increases from 0° to 90° in the case of superior planets viz. Mars, Jupiter and Saturn. On the other hand, in the case of the two interior planets Mercury and Venus 'p' decreases as MK increases from 0° to 90°.

Table 5: *Manda* periphery of Mars in degrees.

MK	*MPH*	*Manda periphery (p)*
15°	2.9	70°.40145
30°	5.7	71°.62831
45°	8.5	75°.52901
60°	10.9	79°.08165
75°	12.4	80°.65991
90°	13	81°.68142

Table 6: The range of *manda* peripheries of other planets.

Planet	*Manda* periphery *(p)*	
	MK (15°)	*MK* (90°)
Mercury	28°.20784	22°.61947
Jupiter	33°.01997	35°.81416
Venus	15°.944	09°.42478
Saturn	46°.25	58°.43363

Table 7: The range of *manda* peripheries of all the planets for MK = 0° and 90° (using equation 7a).

Planet	*Manda* periphery *(p)*	
	MK (0°)	*MK* (90°)
Mars	61°.5752	81°.68142
Mercury	30°.15929	22°.61947
Venus	18°.22124	09°.42478
Jupiter	32°.04424	35°.81416
Saturn	42°.09733	58°.43363

Table 8: The range of *manda* peripheries of all the planets (using equation 7b).

Planet	*Manda* periphery *(p)*	
	MK (0°)	*MK* (90°)
Mars	62°.03774	80°.9824
Mercury	30°.16235	22°.60459
Jupiter	32°.07817	35°.75511
Venus	18°.220618	09°.42370
Saturn	42°.09733	58°.43363

Manda peripheries according to some Indian classical astronomical texts are listed in Table 9, together with our computations for comparison.

From Table 9, it is interesting to note that the same behaviour is seen in the *Āryabhaṭīya* also. In fact, even the ranges of variation of the *manda* periphery as estimated based on the *GL* are close to those of the *Āryabhaṭīya*. However, in

Table 9: Comparison of *manda* peripheries from different texts.

Bodies	Computed Values based on *GL*	The *Āryabhaṭīya*	The *Sūryasiddhānta*
Sun	13°.69 – 14°	13°.5	13°.66 –14°
Moon	31°.59 – 32°.07	31°.5	31°.66 – 32°
Mars	62°.03 – 80°.98	63° – 81°	72° – 75°
Mercury	30°.16 – 22°.60	31°.5 – 22°.5	28° – 30°
Jupiter	32°.07 – 35°.75	31°.5 – 36°.5	32° – 33°
Venus	18°.22 – 09°.42	18° – 9°	11° – 12°
Saturn	42°.09 – 58°.43	40°.5 –58°.5	48° – 49°

Table 10: Discrete values of the equation of the conjunction (*śīghrāṅkas*) of the planets.

Planets	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Mars	58	117	174	228	279	325	365	393	400	368	249	0
Mercury	41	81	117	150	178	199	212	212	195	155	89	0
Jupiter	25	47	68	85	98	106	108	102	89	66	36	0
Venus	63	126	186	246	302	354	402	440	461	443	326	0
Saturn	15	28	39	48	54	57	57	53	45	33	18	0

the case of the Sun and the Moon the peripheries vary as in the *Sūryasiddhānta*.

4 EQUATION OF THE CONJUNCTION OF THE PLANETS

Gaṇeśa Daivajña has provided *śīghrāṅkas* similarly as in the case of the equation of the centre (*mandaphala*) for the convenience of computation. Actual equations of the conjunction (*śīghrāphalas*) are obtained from these *śīghrāṅkas* dividing by 10. The discrete numerical values of *śīghrāṅkas* for the intervals of 15° degrees are listed in Table 10.

In order to determine the *śīghra* peripheries of different planets we adopt the following procedure:

$$\textit{Śīghraphala}\ (SPH) = \sin^{-1}\{[(p/360)\sin(SK)] / \sqrt{[(p/360)^2 \pm 2(p/360)\cos(SK) + 1]}\} \quad (8)$$

where p in the *śīghra* periphery, SPH is the *śīghraphala* and SK is the anomaly of the conjunction (*śīghrakendra*).

Here SK is the anomaly of the conjunction (with the Sun) i.e., SK is the Mean Sun – Mean planet for the superior planets. In the case of Mercury and Venus, SK is the Mean planet – Mean Sun.

Let $(p/360) = r$ (9)

$SPH = \sin^{-1}\{[r\sin(SK)] / \sqrt{[(r)^2 \pm 2(r)\cos(SK) + 1]}\}$ or

Table 11*: Śīghra* periphery of Mars.

SK	*SPH*	*Śīghra* periphery (*p*)
15°	5.8	227°.545
30°	11.7	232°.500
45°	17.4	232°.366
60°	22.8	230°.740
75°	27.9	229°.958
90°	32.5	299°.345
105°	36.5	230°.150
120°	39.3	231°.054
135°	40.0	232°.287
150°	36.8	234°.621
165°	24.9	236°.297
180°	0	0°

$$\sin(SPH) = \{[r\sin(SK)] / \sqrt{[(r)^2 \pm 2(r)\cos(SK) + 1]}\} \quad (10)$$

On squaring both the sides and simplifying equation (10) we get a following equation:

$$r^2\sin^2(SPH) + 2r\cos(SK)\sin^2(SPH) + \sin^2(SPH) - r^2\sin^2(SK) = 0$$

$$[\sin^2(SPH) - \sin^2(SK)]r^2 + 2\cos(SK)\sin^2(SPH)r + \sin^2(SPH) = 0$$

This equation is of the form $Ar^2 + Br + C = 0$, which is a quadratic equation, where $A = [\sin^2(SPH) - \sin^2(SK)]$, $B = 2\cos(SK)\sin^2(SPH)$ and $C = \sin^2(SPH)$.

The roots of a quadratic equation $Ar^2 + Br + C = 0$ are:

$$r = \{-B \pm \sqrt{[B^2 - 4AC]}\} / 2A$$

$$r = \{-B + \sqrt{[B^2 - 4AC]}\} / 2A \quad \text{or}$$

$$r = \{-B - \sqrt{[B^2 - 4AC]}\} / 2A$$

Between these two roots, the valid solution is provided by the equation

$$r = \{-B - \sqrt{[B^2 - 4AC]}\} / 2A$$

From equation (9) we have $p = 360° \times r$.

Thus the *śīghra* periphery

$$p = 360° \times \{-B + \sqrt{[B^2 - 4AC]}\} / 2A \quad (11)$$

Using the above equations we computed the *śīghra* peripheries of Mars and listed the values in Table 11.

From Table 11 as SK varies from 15° to 165° the *śīghra* periphery '*p*' varies from 227°.545 to 236°.297. We express the *śīghra* periphery '*p*' for any given SK in the form

$$p = A + B\sin(SK) \quad (12)$$

To determine A and B we choose, for example SK = 30° and 165°. By solving the linear equations, we obtained A = 240°.372 and B = –15°.7429.

When SK = 0° or 180° equation (12) becomes $p = A$. Hence we can determine the *śīghra* peri-

Table 12: The range of *śīghra* peripheries of all the planets for *MK*= 0° and 90°.

Planet	*Śīghra* periphery (*p*)	
	SK (0°)	*SK* (180°)
Mars	230°.8441	236°.2975
Mercury	133°.0147	137°.4724
Jupiter	68°.1567	72°.55133
Venus	259°.0559	262°.653
Saturn	37°.13942	40°.36791

Table 13 : Comparison of *śīghra* periphery values from different texts.

Planet	Computed Values Based on the *GL*	The *Āryabhaṭīya*	The *SūryaSiddhānta*
Mars	230°.8441 – 236°.2975	229°.5 – 238°.5	232° – 235°
Mercury	133°.0147 – 137°.4724	130°.5 – 139°.5	132° – 133°
Jupiter	68°.1567 – 72°.55133	67°.5 – 72°	72° – 70°
Venus	259°.055 – 262°.653	256°.5 – 265°.5	260° – 262°
Saturn	40°.36791 – 37°.13942	40°.4 – 36°	40° – 39°

pheries of planets from the range of *SK* = 0° to 180° which are listed in Table 12.

The above values of *śīghra* peripheries are compared with other texts to draw a conclusion on our method of computation (see Table 13).

5 CONCLUDING REMARKS

In the above sections we have analyzed the discrete *mandāṅkas* and *śighrāṅkas* given in the *Grahalāghava* of Gaṇeśa Daivajña. We have obtained the ranges of the corresponding *manda peripheries for all bodies* and *śīghra* peripheries for the five planets and compared them with those of the *Āryabhaṭīya* and in the *Sūryasiddhānta*. We find that the ranges of peripheries of planets are closer to those of the *Āryabhaṭīya*, while ranges of *manda* peripheries of the Sun and the Moon vary as in the *Sūryasiddhānta*. However the results obtained are approximate ones; the reasons for this are:

(1) The equation of the centre and the conjunct-tion (*manda* and *śīghraphalas*) given in the *GL* are over wide intervals of 15°; and
(2) The given numerical values are in integers, avoiding fractions in the case of the five planets.

The constants *A* and *B* in equations (7) and (8) obtained are slightly different for different choices of related linear equations. This discrepancy is due to the approximations mentioned above.

6 NOTES

1. Āryabhaṭa I (born 476 C.E) gives, just in one *śloka* (verse), the rule to obtain the *jyā* (*R* sine) of any angle between 0° to 90° at an interval of 3° 45′. He gives the differences between successive values in arc-minutes (*kalās*). Āryabhaṭa's value for the constant co-efficient *R* is 3438′, which is the nearest integer value to a radian.

7 REFERENCES

Primary Sources in Sanskrit

Āryabhaṭiyam of Āryabhaṭa. Edited and translated with notes by K.S. Shukla and K.V. Sarma, INSA, New Delhi, 1976.

Grahalāghavam of Gaṇeśa Daivajña. English exposition, mathematical notes, examples and tables by Dr. S. Balachandra Rao and Dr. S.K. Uma. INSA, New Delhi, 2006.

Grahalāghava of Gaṇeśa Daivajña. With commentary of Viśvanāthadaivajña and Hindi commentary by Kedāradatta Joshī, Motilal Banarasidass. Varanasi, 1981.

Grahalāghava of Gaṇeśa Daivajña. With commentary of Mallāri and Hindi commentary by Ramachandra Pandeya. Chowkamba. Sanskrit Series, Varanasi, 1994.

SūryaSiddhānta. Translation by Reverend E. Burgess, edited and reprinted by Phanindralal Gangooly with an Introduction by P.C. Sengupta. Motilal Banarasidass Publishers Pvt Ltd, Delhi, 1989.

Sūryasiddhānta. Edited with the commentary of Parameśvara by K.S. Shukla. Lucknow, 1957.

Siddhāntaśekhara of Sripati, Edited by Babuāji Miśra, Calcutta University Press, Calcutta, 1934.

Tantrasaṅgraha of NilakaṇṭaSomayāji by K. Ramasubramanian and M.S. Sriram, Hindustan Book Agency (India), New Delhi, 2011.

Secondary Sources in English

Balachandra Rao, S., 2000. *Ancient Indian Astronomy – Planetary Positions and Eclipses*. New Delhi, B.R.P.C. Ltd.

Balachandra Rao, S., 2000. *Indian Astronomy – An Introduction*. Hyderabad, Universities Press (India) Ltd.

Ramasubramanian, K., Srinivas, M.D., and Sriram, M.S, 1994. Modifications of the earlier Indian planetary model by Kerala astronomers (c.1500 AD) and implied heliocentric picture of planetary motion. *Current Science*, 66, 784–790.

Pingree, D., 1978. History of mathematical astronomy in India. In Gillispie, C.C. (ed.). *Dictionary of Scientific Biography. Volume XV, Supplement I*. New York, Charles Scribner's Sons. Pp. 533–633.

Pingree, D., 1981. *Jyotiḥśāstra*. Astral and mathematical literature. In Gonda, J. (ed.). *A History of Indian Literature*. Volume VI, Facs. 4. Wiesbaden, Otto Harrassowitz.

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